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The Marcinkiewicz–Zygmund Inequality on a Smooth Simple Arc

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In this paper, the Marcinkiewicz–Zygmund inequality on a $C^{2+\delta}$ smooth simple arc is obtained. Then we get some results about approximation of the interpolation polynomial on the arc. © 1995 Academic Press, Inc.

1. INTRODUCTION

For $x_k = 2k\pi/(2n+1)$, $k = 0, 1, ..., 2n$, and for arbitrary trigonometric polynomials $T_n(x)$ of degree at most n, the well-known Marcinkiewicz–Zygmund inequality can be written as [11]

$$c_1 \|T_n\|_p \leqslant \left\{\frac{1}{2n+1} \sum_{k=0}^{2n} |T_n(x_k)|^p\right\}^{1/p} \leqslant c_2 \|T_n\|_p, \qquad 1 < p < +\infty,$$

where c_1 and c_2 are positive constants depending only on p.

There are many extension forms of the inequality (see, for example, [4, 6, 9]), and they play the important roles in the research of approximation by interpolation polynomial and the theory of operators. In this paper, we will prove the inequality on any $C^{2+\delta}$ smooth simple arc in the complex plane and get some theorems of approximation by interpolation polynomial on the arc.

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As we know, there are many elegant approximation results on the interval $[-1, 1]$, but there are very few papers discussing polynomial approximation on a simple arc (not a closed Jordan curve) in the complex plane. It is known that approximation polynomials can be constructed by Dzjadyk's kernels on the arc [2]. But the method is very complicated, comparing with construction of interpolation polynomials. One of our theorems shows the mean convergence of the interpolation polynomials on the arc. For uniform approximation, we will give an estimation on the error of the $(n-1)$th interpolation polynomial by the $(n-1)$th best approximation rate timing $\log n$.

THEOREM 1. *Let Γ be a $C^{2+\delta}$ smooth simple arc in the complex plane* $\mathbf{C}$. *There exists $\{z_{k,n}: 0 \leqslant k < n\} \subset \Gamma$ such that for $1 < p < +\infty$ and any algebra polynomial $P_{n-1}(z)$ of degree at most $n-1$,*

$$c_3 \|P_{n-1}\|_{L^p(\Gamma)} \leqslant \left\{ \sum_{k=0}^{n} |P_{n-1}(z_{k,n})|^p \, |z_{k+1,n} - z_{k,n}| \right\}^{1/p} \leqslant c_4 \|P_{n-1}\|_{L^p(\Gamma)} \tag{1.1}$$

where c_3 and c_4 are positive constants depending only on p and Γ, and $z_{n,n} = z_{0,n}$.

For $f \in C(\Gamma)$, let $L_{n-1}(f, z)$ be the $(n-1)$th Langrange interpolation polynomial to $f(z)$ at $\{z_{k,n}: 0 \leqslant k < n\}$. For $f \in L^1(\Gamma)$, let $L_{n-1}^*(f, z)$ denote the polynomial of degree at most $n-1$ and satisfying

$$L_{n-1}^*(f, z_{k,n}) = \frac{1}{|\widetilde{z_{k,n} z_{k+1,n}}|} \int_{z_{k,n}}^{z_{k+1,n}} f(\zeta) \, |d\zeta|, \qquad 0 \leqslant k < n. \tag{1.2}$$

The following two theorems are the applications of Theorem 1.

THEOREM 2. *Under the conditions of Theorem* 1, *there exists $\{z_{k,n}\}$ such that*

$$\lim_{n \to +\infty} \|f(z) - L_{n-1}(f, z)\|_{L^p(\Gamma)} = 0 \tag{1.3}$$

holds for any $f \in C(\Gamma)$ and $0 < p < +\infty$.

THEOREM 3. *Under the conditions of Theorem* 1, *there exists $\{z_{k,n}\}$ such that*

$$\lim_{n \to +\infty} \|f(z) - L_{n-1}^*(f, z)\|_{L^p(\Gamma)} = 0 \tag{1.4}$$

holds for $1 < p < +\infty$ and any $f \in L^p(\Gamma)$.

Concerning uniform approximation of interpolation polynomials, we have the following result.

THEOREM 4. *Under the conditions of Theorem* 1, *there exists* $\{z_{k,n}\}$ *such that for any* $f \in C(\Gamma)$, $n > 1$,

$$\|f(z) - L_{n-1}(f, z)\|_{L^\infty(\Gamma)} \leqslant c_5 \log n E_{n-1}(f)$$

where c_5 *is a positive constant depending only on* Γ *and* $E_{n-1}(f)$ *is the best approximation rate of* f *in the* $L^\infty(\Gamma)$ *norm by polynomials of degree at most* $n-1$.

Without specific declaration, throughout we assume that Γ is a $C^{2+\delta}$ smooth simple arc and $1 < p < +\infty$. We use the notations c and c_j to denote positive constants depending only on Γ and p, and in different places the notation c may not represent the same constant. The notation $A \asymp B$ means

$$cA \leqslant B \leqslant cA.$$

2. SOME COMPLEX GEOMETRY

Let U be the unit disc $\{w : |w| < 1\}$. Let $z = \Psi(w)$ be the conformal map of $\mathbf{C}\backslash\bar{U}$ onto $\mathbf{C}\backslash\Gamma$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Without loss of generality we may further assume $\zeta_0 = \Psi(1)$ and $\zeta_1 = \Psi(e^{i\theta_1})$ $(0 < \theta_1 < 2\pi)$ are the two ends of Γ.

Evidently,

$$z \to 1 \Big/ \sqrt{\frac{z-\zeta_0}{z-\zeta_1}} - 1$$

maps $\mathbf{C}\backslash\Gamma$ onto the exterior region of a smooth Jordan curve. Some elementary calculations can show that the curvature does not jump at the image of each end and the Jordan curve is $C^{2+\delta}$ smooth. As in the Proof of Theorem 3.6 and Theorem 3.9 of [6], we have (see [2, p. 386])

$$\left|\frac{\Psi'(w)}{(1-w^{-1})(1-e^{i\theta_1}w^{-1})}\right| \asymp 1, \qquad |w| \leqslant 1 \tag{2.1}$$

and

$$\left|\frac{d}{dw}\left[\frac{\Psi'(w)}{(1-w^{-1})(1-e^{i\theta_1}w^{-1})}\right]\right| \leqslant c.$$

Then we have

$$|\Psi''(w)| \leqslant c_6. \tag{2.2}$$

For $w_1 = \rho_1 e^{it_1}$, $w_2 = \rho_2 e^{it_2}$, $1 \leqslant \rho_1, \rho_2 \leqslant 2$, when both t_1 and t_2 are in the same interval $[0, \theta_1]$ or $[\theta_1, 2\pi]$, we have

$$|\Psi(w_1) - \Psi(w_2)| \asymp |w_1 - w_2| \min_{\theta^* = 0, \theta_1} (|w_1 - e^{\theta^*}| + |w_1 - w_2|).$$

For simplicity of notation, we always assume the minimum of the above formula is obtained at $\theta^* = 0$; then

$$|\Psi(w_1) - \Psi(w_2)| \asymp |w_1 - w_2| (|w_1 - 1| + |w_1 - w_2|). \tag{2.3}$$

Since

$$\tfrac{1}{2}(|w_1 - 1| + |w_2 - 1|) \leqslant |w_1 - 1| + |w_1 - w_2| \leqslant 2(|w_1 - 1| + |w_2 - 1|),$$

we also have

$$|\Psi(w_1) - \Psi(w_2)| \asymp |w_1 - w_2| (|w_1 - 1| + |w_2 - 1|). \tag{2.4}$$

When Ψ is restricted on $\{e^{i\theta}: \theta \in [0, \theta_1]\}$, it is an isomorphic map onto Γ. We denote the inverse map by Φ_0. For $t \in [\theta_1, 2\pi]$, let

$$J(e^{it}) = \Phi_0(\Psi(e^{it})).$$

Then $J: \{e^{it}: t \in [\theta_1, 2\pi]\} \to \{e^{i\theta}: \theta \in [0, \theta_1]\}$ is an isomorphic map. We denote the inverse map by J^{-1}. For $\theta \in [0, \theta_1]$, set

$$J(e^{i\theta}) = J^{-1}(e^{i\theta}).$$

Then J is an inverse direction isomorphic of unit circle ∂U with 1 and $e^{i\theta_1}$ both fixed points. Evidently

$$\Psi(e^{it}) = \Psi(J(e^{it})), \qquad t \in [0, 2\pi]. \tag{2.5}$$

LEMMA 2.1. *For $t_1, t_2 \in [0, \theta_1]$, $1 \leqslant \rho \leqslant 2$, we have*

$$|J(e^{it_1}) - J(e^{it_2})| \asymp |e^{it_1} - e^{it_2}| \tag{2.6}$$

and

$$|\Psi(\rho e^{it_1}) - \Psi(e^{it_2})| \asymp |\Psi(\rho J(e^{it_1})) - \Psi(J(e^{it_2}))|. \tag{2.7}$$

Proof. By (2.3) and (2.5), we have

$$\begin{aligned}|e^{it_j}-1| &\asymp |\Psi(e^{it_j})-\Psi(1)|^{1/2}\\ &= |\Psi(J(e^{it_j}))-\Psi(1)|^{1/2}\\ &\asymp |J(e^{it_j})-1|, \qquad j=1,2.\end{aligned}$$

Togther with (2.4) and (2.5)

$$\begin{aligned}|e^{it_1}-e^{it_2}|\,(|e^{it_1}-1|+|e^{it_2}-1|) &\asymp |\Psi(e^{it_1})-\Psi(e^{it_2})|\\ &= |\Psi(J(e^{it_1}))-\Psi(J(e^{it_2}))|\\ &\asymp |J(e^{it_1})-J(e^{it_2})|\,(|J(e^{it_1})-1|+|J(e^{it_2})-1|)\\ &\asymp |J(e^{it_1})-J(e^{it_2})|\,(|e^{it_1}-1|+|e^{it_2}-1|).\end{aligned}$$

Then we get (2.6).

By (2.4) and (2.6) we have

$$\begin{aligned}|\Psi(\rho e^{it_1})-\Psi(e^{it_2})| &\asymp ((\rho-1)+|e^{it_1}-e^{it_2}|)((\rho-1)+|e^{it_1}-1|+|e^{it_2}-1|)\\ &\asymp ((\rho-1)+|J(e^{it_1})-J(e^{it_2})|)((\rho-1)+|J(e^{it_1})-1|\\ &\quad+|J(e^{it_2})-1|)\\ &\asymp |\Psi(\rho J(e^{it_1}))-\Psi(J(e^{it_2}))|\\ &= |\Psi(\rho J(e^{it_1}))-\Psi(e^{it_2})|.\end{aligned}$$

Then we have (2.7).

Let

$$\Gamma_n=\left\{\Psi\left[\left(1+\frac{1}{n}\right)e^{it}\right]: t\in[0,2\pi)\right\}$$

and

$$\Gamma_{1n}=\left\{\Psi\left[\left(1+\frac{1}{n}\right)e^{it}\right]: t\in[0,\theta_1]\right\}$$

$$\Gamma_{2n}=\left\{\Psi\left[\left(1+\frac{1}{n}\right)e^{it}\right]: t\in[\theta_1,2\pi]\right\}.$$

Obviously, for Γ_{1n} or Γ_{2n}, the ratio of arclength of any subarc to the chord length is uniformly bounded (this argument can also be proved strictly by (2.1) and (2.3)).

Let

$$U(z,r)=\{\zeta\in\mathbf{C}: |\zeta-z|<r\}, \tag{2.8}$$

then for any $z \in \mathbf{C}$, we have

$$\int_{\Gamma_{jn} \cap U(z, r)} |d\zeta| \leqslant cr, \qquad j = 1, 2.$$

Hence

$$\int_{\Gamma_n \cap U(z, r)} |d\zeta| \leqslant cr. \tag{2.9}$$

Then $\{\Gamma_n\}$ is a series of uniformly regular Jordan curves [1].

3. Cauchy's Integral Operator

For $f \in L^1(\Gamma_n)$, let

$$T_n f(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{3.1}$$

For $z \in \Gamma_n$ we define $T_n f(z)$ by the non-tangent limit of the above formula.

Lemma 3.1. *Suppose* $1 < p < \infty$, $f \in L^p(\Gamma_n)$, *then*

$$\left\{\int_{\Gamma_n} |T_n f(z)|^p\, |dz|\right\}^{1/p} \leqslant c \left\{\int_{\Gamma_n} |f(\zeta)|^p\, |d\zeta|\right\}^{1/p}. \tag{3.2}$$

Furthermore, for any positive measure μ, *we have*

$$\left\{\int \frac{1}{1 + \bar{\mu}(z)}\, |T_n f(z)|^p\, |d\mu(z)|\right\}^{1/p} \leqslant c \left\{\int_{\Gamma_n} |f(\zeta)|^p\, |d\zeta|\right\}^{1/p} \tag{3.3}$$

where

$$\bar{\mu}(z) = \sup_{r > 0} \frac{\mu(U(z, r))}{r}. \tag{3.4}$$

This lemma is a corollary of Proposition 6 in [1], since Γ_n is a series of uniformly regular Jordan curves.

4. Adjusted Fejer's Points

For $n > 0$, let

$$z^*_{k,n} = \Psi(e^{i(2k\pi/n)}), \qquad k = 0, 1, ..., n-1. \tag{4.1}$$

They are so-called Fejer's points. It is well-known that the interpolation polynomials at Fejer's points have good property of approximation in

Jordan domains. Since Ψ maps two points $\{e^{it}, J(e^{it})\}$ to the same point on Γ, some Fejer's points may be very close and even meet each other. We can see that if a pair of Fejer's points meet each other, then we need multiple interpolations at the point, which means interpolating to the derivative of the function. Under the assumption $f(z) \in C(\Gamma)$, the derivative may not exist. In this paper, we adjust some Fejer's points in order to keep distance between them.

Let

$$K_{1n} = \left\{ k : 0 \leqslant k < n, \frac{2k\pi}{n} \in [0, \theta_1] \right\}$$

$$K_{2n} = \left\{ k : 0 \leqslant k < n, \frac{2k\pi}{n} \in [\theta_1, 2\pi) \right\}.$$

By Lemma 2.1, we have

$$|\arg(J(e^{it_1})) - \arg(J(e^{it_2}))| \geqslant c_7 \, |t_1 - t_2|, \qquad t_1, t_2 \in (\theta_1, 2\pi). \tag{4.2}$$

Let

$$c_0 = \min \left\{ \frac{c_7 \pi}{2}, \frac{\pi}{2} \right\}. \tag{4.3}$$

Now we adjust some Fejer's points as follows.

(1) For $k \in K_{2n}$, let

$$\lambda_{k,n} = \arg(J(e^{i(2k\pi)/n})). \tag{4.4}$$

(2) For $j \in K_{1n}$, a function flag(j) is defined by

$$\text{flag}(0) = 0$$

and

$$\text{flag}(j) = \begin{cases} \text{flag}(j-1), & \text{if } \min_{k \in K_{2n}} \left| \frac{2(j-1)\pi}{n} - \lambda_{k,n} \right| \geqslant \frac{c_0}{2n}, \\ \text{flag}(j-1) + 1, & \text{otherwise.} \end{cases} \tag{4.5}$$

(3) For $j \in K_{1n}$, let

$$\lambda_{j,n} = \begin{cases} \frac{2j\pi}{n}, & \text{if } \min_{k \in K_{2n}} \left| \frac{2j\pi}{n} - \lambda_{k,n} \right| \geqslant \frac{c_0}{2n}, \\ \frac{2j\pi}{n} + (-1)^{\text{flag}(j)} \frac{c_0}{n}, & \text{otherwise.} \end{cases} \tag{4.6}$$

It is necessary to point out that for $j_0=[\theta_1 n/2\pi]$, (4.6) may cause

$$\lambda_{j_0,n}\notin[0,\theta_1]; \tag{4.7}$$

in this case, we define

$$\lambda_{j_0,n}=\frac{2j\pi}{n}-\frac{c_0}{n}.$$

By (4.4) and (4.6), we have $\lambda_{k,n}\in[0,\theta_1]$, $k=0,1,\ldots,n-1$.

(4) Let

$$z_{k,n}=\Psi(e^{i\lambda_{k,n}}),\qquad k=0,1,\ldots,n-1; \tag{4.8}$$

these are the so-called adjusted Fejer's points.

LEMMA 4.1. *For $j\neq k$, we have*

$$|\lambda_{j,n}-\lambda_{k,n}|\geqslant\frac{c_0}{2n}. \tag{4.9}$$

Proof. For $j,k\in K_{2n}$, (4.2) implies (4.9). For $j,k\in K_{1n}$, by (4.3) we have

$$\begin{aligned}|\lambda_{j,n}-\lambda_{k,n}|&\geqslant\left|\frac{2j\pi}{n}-\frac{2k\pi}{n}\right|-\left|\lambda_{j,n}-\frac{2j\pi}{n}\right|-\left|\lambda_{k,n}-\frac{2k\pi}{n}\right|\\&\geqslant\frac{2\pi}{n}-\frac{2c_0}{n}\geqslant\frac{2c_0}{n}.\end{aligned}$$

When $j\in K_{1n}$ and $k\in K_{2n}$, by (4.6) we have (4.9) too.

5. UNIFORMLY SEPARATED

A set $\{w_j\}\subset U$ is called ε-uniformly separated if

$$\inf_k\prod_{j\neq k}\left|\frac{w_j-w_k}{1-\overline{w_k}w_j}\right|\geqslant\varepsilon>0, \tag{5.1}$$

and $\{w_j\}\subset U$ is called ε-weakly separated if

$$\inf_{j\neq k}\left|\frac{w_j-w_k}{1-\overline{w_k}w_j}\right|\geqslant\varepsilon>0. \tag{5.2}$$

Let $D_n=\operatorname{int}(\Gamma_n)$, $\phi_n\colon D_n\to U$ be a conformal map.

LEMMA 5.1. *For any* $n>0$, $\{\phi_n(z_{k,n}): 0 \leqslant k < n\}$ *is* c_8*-weakly separated.*

Proof. For $j \neq k$, from (2.3) and (4.9) we have

$$|z_{j,n} - z_{k,n}| = |\Psi(e^{i\lambda_{j,n}}) - \Psi(e^{i\lambda_{k,n}})|$$
$$\asymp |\lambda_{j,n} - \lambda_{k,n}| \, (|\lambda_{j,n} - \lambda_{k,n}| + |\lambda_{j,n}|)$$
$$\geqslant \frac{c_0}{2n}\left(\frac{c_0}{2n} + |\lambda_{j,n}|\right). \tag{5.3}$$

Since

$$d(z_{j,n}, \Gamma_n) \asymp \left|\Psi\left[\left(1+\frac{1}{n}\right)e^{i\lambda_{j,n}}\right] - \Psi(e^{i\lambda_{j,n}})\right|$$
$$\asymp \frac{1}{n}\left(\frac{1}{n} + |\lambda_{j,n}|\right), \tag{5.4}$$

we have

$$|z_{j,n} - z_{k,n}| \geqslant c_9 \, d(z_{j,n}, \Gamma_n). \tag{5.5}$$

As in the proof of Lemma 1 in [7], we know that $\{\phi_n(z_{k,n}): 0 \leqslant k < n\}$ is $(c_9/16)$-weakly separated.

In fact, we will show that $\{\phi_n(z_{k,n}): 0 \leqslant k < n\}$ is c-uniformly separated. Let δ_w denote the Dirac mass at w. It is sufficient to show that

$$v_n = \sum_{k=0}^{n-1} \delta_{\phi_n(z_{k,n})}(1 - |\phi_n(z_{k,n})|^2) \tag{5.6}$$

is a Carleson measure [3].

LEMMA 5.2. *For any* $f \in E^p(D_n)$, *we have*

$$\|f\|_{L^p(\Gamma)} \leqslant c \, \|f\|_{L^p(\Gamma_n)} \tag{5.7}$$

and

$$\sum_{k=0}^{n-1} |f(z_{k,n})|^p \, d_k \leqslant c_{10} \, \|f\|^p_{L^p(\Gamma_n)} \tag{5.8}$$

where $d_k = d(z_{k,n}, \Gamma_n)$.

Proof. Since $f(z)$ is analytic in D_n, $T_n f(z) = f(z)$. For $d\mu$, the arc length measure of Γ, from (3.4) we have $\bar{\mu}(z) \leqslant c$. By (3.3) we have (5.6).

Now we are going to show that

$$\sum_{k \in K_{1n}} |f(z_{k,n})|^p \, d_k \leqslant c_{10} \, \|f\|^p_{L^p(\Gamma_n)}. \tag{5.8'}$$

The other part of the sum for $k \in K_{2n}$ can be estimated in the same way.

By (5.5) we have

$$|z_{j,n} - z_{k,n}| > \frac{c_9}{2} (d_j + d_k). \tag{5.9}$$

Let $r_k = (c_9/8) \, d_k$ and let $U_k = U(z_{k,n}, r_k)$, then

$$d(U_j, U_k) \geqslant 2(r_j + r_k). \tag{5.10}$$

Let $V = \bigcup_{k \in K_{1n}} \partial U_k$ and let σ be the arc length measure of V. Since $|f(z)|^p$ is subharmonic on D_n, we have

$$\sum_{k \in K_{1n}} |f(z_{k,n})|^p \, d_k \leqslant c \int |f|^p \, d\sigma. \tag{5.11}$$

In order to use (3.3) to the measure σ, we have to estimate

$$\bar{\sigma}(z) = \sup_{r > 0} \frac{\sigma(U(z, r))}{r}, \qquad z \in V.$$

For $z \in \partial U_k$, if $r \leqslant 2r_k$, by (5.10) we have $U(z, r) \cap V = U(z, r) \cap \partial U_k$, then

$$\sup_{0 < r \leqslant 2r_k} \frac{\sigma(U(z, r))}{r} = \tau. \tag{5.12}$$

When $r > 2r_k$, by (5.4) we have

$$r \geqslant c d_k \geqslant \frac{c_{11}(k+1)}{n^2}. \tag{5.13}$$

If $U(z, r) \cap \partial U_j \neq \phi$ and $j \neq k$, by (5.10) we have $r \geqslant 2(r_j + r_k)$. Then $|z_{j,n} - z_{k,n}| \leqslant (r_j + r_k) + r < 2r$. This implies $z_{j,n} \in U(z_{k,n}, 2r)$. Then

$$\sigma(U(z, r)) \leqslant \sum_{|z_{j,n} - z_{k,n}| < 2r} |\partial U_j| = 2\pi \sum_{|z_{j,n} - z_{k,n}| < 2r} r_j. \tag{5.14}$$

By (2.4) we have

$$\{ j \in K_{1n} \colon |z_{j,n} - z_{k,n}| < 2r \} \subset \{ j \in K_{1n} \colon |k - j| (k + j) < c_{12} n^2 r \}.$$

Let $S(k, r)$ denote the set in the right side of the above formula, and let j_1, j_2 be the smallest and largest integer in $S(k, r)$ respectively. By (5.4) and (5.14),

$$\sigma(U(z, r)) \leqslant c \sum_{S(k, r)} \frac{j+1}{n^2} = c \frac{(j_2 - j_1 + 1)(j_2 + j_1 + 2)}{2n^2}$$

$$= c \frac{j_2^2 - j_1^2 + 3j_2 + j_1 + 2}{2n^2}. \tag{5.15}$$

Noting (5.13), because of $k^2 - j_1^2 < c_{12} n^2 r$ and $j_2^k - k^2 < c_{12} n^2 r$, we have

$$j_2^2 - j_1^2 < 2c_{12} n^2 r$$

and

$$3j_1 + j_2 + 2 \leqslant 3(j_2^2 - j_1^2) + 4(k+2) \leqslant 6c_{12} n^2 r + \frac{4n^2 r}{c_{11}}.$$

From (5.15), we have

$$\sigma(U(z, r)) \leqslant cr, \qquad r > 2r_k.$$

Togther with (5.12) we have $\bar{\sigma}(z) \leqslant c$. By (3.3) and (5.11) we have (5.8′).

Using Lemma 5.2, we can get the following statement similar to the Proof of Lemma 3 in [7].

Lemma 5.3. *For any* $n > 0$, $\{\phi_n(z_{k,n}) : 0 \leqslant k < n\}$ *is c-uniformly separated.*

Since $\{\phi_n(z_{k,n}) : 0 \leqslant k < n\}$ is c-uniformly separated, we can use the free interpolation theory to find an analytic function satisfying the interpolation condition at $\{\phi_n(z_{k,n})\}$.

Lemma 5.4. *Given complex numbers* a_k, $k = 0, 1, ..., n-1$, *there exist* $g(z) \in E^p(D_n)$ *and* $h(z) \in H^\infty(D_n)$ *such that*

$$g(z_{k,n}) = h(z_{k,n}) = a_k, \qquad k = 0, 1, ..., n-1,$$

and

$$\|g\|_{L^p(\Gamma_n)} \leqslant c \left\{ \sum_{k=0}^{n-1} |a_k|^p \, d_k \right\}^{1/p}$$

$$\|h\|_{L^\infty(\Gamma_n)} \leqslant c \max_{0 \leqslant k < n} |a_k|.$$

Proof. The proof is the same as the Proof of Lemma 4 in [7].

6. An Estimation of $|\omega_n(z)|$ on Γ_n

Let

$$\omega(z)=\prod_{k=0}^{n-1}(z-z_{k,n}). \tag{6.1}$$

Lemma 6.1. *For $z\in\Gamma_n$, we have*

$$|\omega(z)|\asymp[\Psi'(\infty)]^n. \tag{6.2}$$

Proof. Let

$$\omega^*(z)=\prod_{k=0}^{n-1}(z-z^*_{k,n}) \tag{6.3}$$

where $z^*_{k,n}$ are Fejer's points.

As in the Proof of Lemma 4 in [7] (taking $\alpha=2$), we have

$$|\omega^*(z)|\asymp[\Psi'(\infty)]^n, \qquad z\in\Gamma_n.$$

Therefore it is sufficiently to show that

$$\left|\frac{\omega(z)}{\omega^*(z)}\right|\asymp 1. \tag{6.4}$$

Obviously,

$$\frac{\omega(z)}{\omega^*(z)}=\prod_{k\in S_n}\frac{z-\Psi(e^{i\lambda_{k,n}})}{z-\Psi(e^{i(2k\pi)/n})} \tag{6.5}$$

where

$$S_n=\left\{k\in K_{1n}: \lambda_{k,n}\neq\frac{2k\pi}{n}\right\}. \tag{6.6}$$

Let $z=\Psi[(1+1/n)e^{i\theta}]\in\Gamma_n$. We will show (6.4) in the cases $\theta\in[0,\theta_1]$ and $\theta\in(\theta_1,2\pi)$ separately.

Case 1. $\theta\in[0,\theta_1]$, by (2.4) we have

$$|z-z^*_k|\asymp\left(\frac{1}{n}+\left|\theta-\frac{2k\pi}{n}\right|\right)\left(\frac{1}{n}+|\theta|+\left|\frac{2k\pi}{n}\right|\right)\asymp|z-z_k|. \tag{6.7}$$

Therefore it will not cause any essential change in estimating (6.4) if we omit several factors in the right-side product of (6.5). Hence we may assume that (3.6) does not happen and the elements in S_n range as $k(0)<k(1)<\cdots<k(s)$. By (4.5) we have

$$\operatorname{flag}(k(j))=j, \qquad j=0,1,\ldots,s. \tag{6.8}$$

Evidently,

$$[0,\theta_1]=\left[0,\frac{2k(0)\pi}{n}\right]\cup\left[\frac{2k(s)\pi}{n},\theta_1\right]\cup_{l=0}^{s-1}\left[\frac{2k(l)\pi}{n},\frac{2k(l+1)\pi}{n}\right].$$

When $\theta\in[2k(l)\pi/n, 2k(l+1)\pi/n]$, by (6.6) and (6.7) we have

$$\begin{aligned}\log\left|\frac{\omega(z)}{\omega^*(z)}\right| &= \sum_{j=1}^{[(l-1)/2]}\log\left|\frac{(z-z_{k(l-2j),n})(z-z_{k(l-2j-1),n})}{(z-z^*_{k(l-2j),n})(z-z^*_{k(l-2j-1),n})}\right| \\ &\quad\times\sum_{j=1}^{[(s-l-1)/2]}\log\left|\frac{(z-z_{k(l+2j),n})(z-z_{k(l+2j+1),n})}{(z-z^*_{k(l+2j),n})(z-z^*_{k(l+2j+1),n})}\right|+O(1) \\ &= I_1+I_2+O(1).\end{aligned} \tag{6.9}$$

Evidently,

$$\begin{aligned}&\frac{(z-z_{k(l-2j-1),n})(z-z_{k(l-2j),n})}{(z-z^*_{k(l-2j-1),n})(z-z^*_{k(l-2j),n})} \\ &\quad=1-\frac{z_{k(l-2j),n}+z_{k(l-2j-1),n}-z^*_{k(l-2j),n}-z^*_{k(l-2j-1),n}}{z-z^*_{k(l-2j-1),n}} \\ &\quad\quad+\frac{(z^*_{k(l-2j),n}-z_{k(l-2j),n})(z^*_{k(l-2j),n}-z_{k(l-2j-1),n})}{(z-z^*_{k(l-2j),n})(z-z^*_{k(l-2j-1),n})}.\end{aligned} \tag{6.10}$$

Writing $\eta=(-1)^l(c_0/n)$, by (4.6), (4.8), and (6.8) we have

$$\begin{aligned}&|z_{k(l-2j),n}+z_{k(l-2j-1),n}-z^*_{k(l-2j),n}-z^*_{k(l-2j-1),n}| \\ &\quad=|\Psi(e^{i(2k(l-2j)\pi/n)+i\eta})-\Psi(e^{i(2k(l-2j)\pi/n)}) \\ &\quad\quad+\Psi(e^{i(2k(l-2j-1)\pi/n)-i\eta})-\Psi(e^{i(2k(l-2j-1)\pi/n)})| \\ &\quad=\left|\int_0^\eta[\Psi'(e^{i((2k(l-2j)\pi/n)+t)})\,e^{i((2k(l-2j)\pi)/n+t)}\right. \\ &\quad\quad\left.-\Psi'(e^{i((2k(l-2j-1)\pi/n)-t)})\,e^{i((2k(l-2j-1)\pi/n)-t)}]\,dt\right|\end{aligned}$$

By (2.1) and (2.2),

$$\begin{aligned}|z_{k(l-2j),n}+z_{k(l-2j-1),n}-z^*_{k(l-2j),n}-z^*_{k(l-2j-1),n}| &\leqslant c\,\frac{k(l-2j)-k(l-2j-1)}{n}\,\eta \\ &\leqslant c\,\frac{k(l-2j)-k(l-2j-1)}{n^2}. \end{aligned} \tag{6.11}$$

By (6.7) we have

$$\begin{aligned}&\sum_{j=1}^{[(l-1)/2]}\frac{|z_{k(l-2j),n}+z_{k(l-2j-1),n}-z^*_{k(l-2j),n}-z^*_{k(l-2j-1),n}|}{|z-z^*_{k(l-2j-1),n}|}\\ &\quad\leqslant c\sum_{j=1}^{[(l-1)/2]}\frac{k(l-2j)-k(l-2j-1)}{(n\theta-2k(l-2j-1)\pi+1)^2}\\ &\quad\leqslant c\sum_{j=1}^{[(l-1)/2]}\frac{k(l-2j)-k(l-2j-1)}{(n\theta-2k(l-2j-1)\pi+1)(n\theta-2k(l-2j)\pi+1)}\\ &\quad\leqslant c\sum_{m=1}^{l-1}\frac{k(l-m)-k(l-m-1)}{(n\theta-2k(l-m-1)\pi+1)(n\theta-2k(l-m)\pi+1)}\\ &\quad=\frac{c}{2\pi}\left(\frac{1}{n\theta-2k(l-1)\pi+1}-\frac{1}{n\theta-2k(0)\pi+1}\right)\leqslant c.\end{aligned} \tag{6.12}$$

By (2.4) we have

$$\begin{aligned}&\sum_{j=1}^{[(l-1)/2]}\frac{|(z^*_{k(l-2j),n}-z_{k(l-2j),n})(z^*_{k(l-2j),n}-z_{k(l-2j-1),n})|}{|(z-z^*_{k(l-2j),n})(z-z^*_{k(l-2j-1),n})|}\\ &\quad\asymp\sum_{j=1}^{[(l-1)/2]}\frac{(1/n)(k(l-2j)+1)/n}{(|(2k(l-2j)\pi)/n-\theta|+1/n)(|(2k(l-2j)\pi)/n|+|\theta|+1/n)}\\ &\qquad\times\frac{((k(l-2j)-k(l-2j-1)/n)(k(l-2j)/n))}{(|(2k(l-2j-1)\pi)/n-\theta|+1/n)(|(2k(l-2j-1)/n)\pi|+|\theta|+1/n)}\\ &\quad\asymp\sum_{j=1}^{[(l-1)/2]}\frac{k(l-2j)-k(l-2j-1)}{(n\theta-2k(l-2j-1)\pi+1)(n\theta-2k(l-2j)\pi+1)}\leqslant c.\end{aligned} \tag{6.13}$$

Then we have

$$|I_1|\leqslant c,\qquad \theta\in\left[\frac{2k(l)\pi}{n},\frac{2k(l+1)\pi}{n}\right]. \tag{6.14}$$

In the same way, we can get $|I_2| \leqslant c$. Together with (6.9) we have (6.4) for $\theta \in [2k(l)\,\pi/n, 2k(l+1)\,\pi/n]$.

For $\theta \in [0, 2k(0)\,\pi/n] \cup [2k(s)\,\pi/n, \theta_1]$, the procedure of the proof is similar. Then we have (6.4) for $\theta \in [0, \theta_1]$.

Case 2. For $\theta \in (\theta_1, 2\pi)$, let

$$\theta' = \arg(J(e^{i\theta})) \in [0, \theta_1]$$

and let

$$z' = \Psi\left[\left(1+\frac{1}{n}\right)e^{i\theta'}\right].$$

By (2.6), we have

$$|z - z_{k,n}| \asymp |z' - z_{k,n}| \asymp |z' - z^*_{k,n}| \asymp |z - z_{k,n}|. \tag{6.7'}$$

This corresponds to (6.7) in Case 1.

For $\theta' \in [2k(l)\,\pi/n, 2k(l+1)\,\pi/n]$, by (6.7′) we also have formulas such as (6.9) and (6.10). Since $|z' - z^*_{k,n}| \asymp |z - z^*_{k,n}|$, by (6.12) we have

$$\sum_{j=1}^{[(l-1)/2]} \frac{|z_{k(l-2j),n} + z_{k(l-2j-1),n} - z^*_{k(l-2j),n} - z^*_{k(l-2j-1),n}|}{|z - z^*_{k(l-2j-1),n}|}$$
$$\asymp \sum_{j=1}^{[(l-1)/2]} \frac{|z_{k(l-2j),n} + z_{k(l-2j-1),n} - z^*_{k(l-2j),n} - z^*_{k(l-2j-1),n}|}{|z' - z^*_{k(l-2j-1),n}|} \leqslant c,$$

and by (6.13)

$$\sum_{j=1}^{[(l-1)/2]} \frac{|(z^*_{k(l-2j),n} - z_{k(l-2j),n})(z^*_{k(l-2j),n} - z_{k(l-2j-1),n})|}{|(z - z^*_{k(l-2j),n})(z - z^*_{k(l-2j-1),n})|}$$
$$\asymp \sum_{j=1}^{[(l-1)/2]} \frac{|(z^*_{k(l-2j),n} - z_{k(l-2j),n})(z^*_{k(l-2j),n} - z_{k(l-2j-1),n})|}{|z(' - z^*_{k(l-2j),n})(z' - z^*_{k(l-2j-1),n})|} \leqslant c.$$

Then we have (6.4) for $\theta' \in [2k(l)\,\pi/n, 2k(l+1)\,\pi/n]$.

For $\theta' \in [0, 2k(0)\,\pi/n] \cup [2k(s)\,\pi/n, \theta_1]$, the procedure of the proof is similar. Then we have (6.4) for $\theta \in (\theta_1, 2\pi)$.

7. Proof of The Theorems

Proof of Theorem 1. By the Bernstein theorem, we have $\|p_{n-1}\|_{L^p(\Gamma_n)} \leqslant c\,\|p_{n-1}\|_{L^p(\Gamma)}$. For $\{z_{k,n}\}$, the adjusted Fejer's points, from (5.8) we have

$$\sum_{k=0}^{n-1} |P_{n-1}(z_{k,n})|^p\, d_k \leqslant c\,\|P_{n-1}\|^p_{L^p(\Gamma)}.$$

By (2.4),

$$|z_{k+1,n}-z_{k,n}| \asymp d_k. \tag{7.1}$$

The left side of (1.1) holds.

On the other hand, from Lemma 5.4 there exists $g(z) \in E^p(D_n)$ such that

$$g(z_{k,n}) = P_{n-1}(z_{k,n}), \qquad k=0, 1, ..., n-1, \tag{7.2}$$

and

$$\begin{aligned} \|g\|_{L^p(\Gamma_n)} &\leqslant c \left\{ \sum_{k=0}^{n-1} |P_{n-1}(z_{k,n})|^p \, d_k \right\}^{1/p} \\ &\asymp \left\{ \sum_{k=0}^{n-1} |P_{n-1}(z_{k,n})|^p \, |z_{k+1,n}-z_{k,n}| \right\}^{1/p}. \end{aligned} \tag{7.3}$$

By (7.2) we have $P_{n-1}(z) = L_{n-1}(g, z)$, hence

$$\begin{aligned} g(z) - P_{n-1}(z) &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\omega(z)}{\omega(\zeta)} \frac{g(\zeta)}{\zeta - z} \, d\zeta \\ &= \omega(z) \, T_n\left(\frac{g}{\omega}\right)(z), \qquad z \in D_n. \end{aligned}$$

From Lemma 3.1 and Lemma 6.1,

$$\begin{aligned} \|g - P_{n-1}\|_{L^p(\Gamma_n)} &\leqslant \max_{z \in \Gamma_n} |\omega(z)| \left\| T_n\left(\frac{g}{\omega}\right)(z) \right\|_{L^p(\Gamma_n)} \\ &\leqslant c \max_{z \in \Gamma_n} |\omega(z)| \left\| \frac{g}{\omega}(\zeta) \right\|_{L^p(\Gamma_n)} \\ &\leqslant c \max_{z, \zeta \in \Gamma_n} \frac{|\omega(z)|}{|\omega(\zeta)|} \|g\|_{L^p(\Gamma_n)} \\ &\leqslant c \, \|g\|_{L^p(\Gamma_n)}. \end{aligned}$$

Hence

$$\|P_{n-1}\|_{L^p(\Gamma_n)} \leqslant c \, \|g\|_{L^p(\Gamma_n)}.$$

Together with (7.3), the right side of (1.1) holds.

Proof of Theorem 2. It is sufficient to prove the theorem for $1 < p < +\infty$. Let $\{z_{k,n}\}$ be the adjusted Fejer's points. Since $L_{n-1}(f, z)$ is a polynomial of degree at most $n-1$, by (1.1) we have

$$\|L_{n-1}(f, z)\|_{L^p(\Gamma)} \leqslant c \left\{ \sum_{k=0}^{n-1} |f(z_{k,n})|^p \, |z_{k+1,n} - z_{k,n}| \right\}^{1/p}$$
$$\leqslant c \, \|f\|_{L^\infty(\Gamma)}. \tag{7.4}$$

That means $L_{n-1}: C(\Gamma) \to L^p(\Gamma)$ is uniformly bounded; then we have (1.3).

Proof of Theorem 3. First, we show (1.4) for $f \in C(\Gamma)$. By the uniform continuity of $f(z)$ on Γ, we have

$$\lim_{n\to\infty} \sup_{0 \leqslant k < n} \left| f(z_{k,n}) - \frac{1}{|\widehat{z_{k,n} z_{k+1,n}}|} \int_{z_{k,n}}^{z_{k+1,n}} f(\zeta) \, |d\zeta| \right| = 0.$$

Similar to (7.4), we have

$$\lim_{n\to\infty} \|L_{n-1}(f, z) - L^*_{n-1}(f, z)\|_{L^p(\Gamma)}$$
$$\leqslant c \lim_{n\to\infty} \sup_{0 \leqslant k < n} \left| f(z_{k,n}) - \frac{1}{|\widehat{z_{k,n} z_{k+1,n}}|} \int_{z_{k,n}}^{z_{k+1,n}} f(\zeta) \, |d\zeta| \right|$$

From Theorem 2 we have (1.4) for $f \in C(\Gamma)$.

For $f(z) \in L^p(\Gamma)$, by Theorem 1 and Hölder's inequality

$$\|L^*_{n-1}(f, z)\|^p_{L^p(\Gamma)} \leqslant c \sum_{k=0}^{n-1} \left| \frac{1}{|\widehat{z_{k,n} z_{k+1,n}}|} \int_{z_{k,n}}^{z_{k+1,n}} f(\zeta) \, |d\zeta| \right|^p |z_{k+1,n} - z_{k,n}|$$
$$\leqslant c \sum_{k=0}^{n-1} \frac{|z_{k+1,n} - z_{k,n}|}{|\widehat{z_{k,n} z_{k+1,n}}|} \int_{z_{k,n}}^{z_{k+1,n}} |f(\zeta)|^p \, |d\zeta|$$
$$\leqslant c_{13} \, \|f\|^p_{L^p(\Gamma)}.$$

That means $L^*_{n-1}: L^p(\Gamma) \to L^p(\Gamma)$ uniformly bounded. Since $C(\Gamma)$ is dense in $L^p(\Gamma)$, we have

$$\overline{\lim_{n\to\infty}} \|f(z) - L^*_{n-1}(f, z)\|_{L^p(\Gamma)}$$
$$\leqslant \inf_{h \in C} \overline{\lim_{n\to\infty}} \left(\|f(z) - h(z) - L^*_{n-1}(f-h, z)\|_{L^p(\Gamma)} \right.$$
$$\left. + \|h(z) - L^*_{n-1}(h, z)\|_{L^p(\Gamma)} \right)$$
$$\leqslant \inf_{h \in C(\Gamma)} \left\{ (1 + c_{13}) \, \|f(z) - h(z)\|_{L^p(\Gamma)} \right.$$
$$\left. + \overline{\lim_{n\to\infty}} \|h(z) - L^*_{n-1}(h, z)\|_{L^p(\Gamma)} \right\} = 0.$$

Proof of Theorem 4. In fact, it is sufficient to show

$$\|P_{n-1}(z)\|_{L^\infty(\Gamma)} \leqslant c \log n \max_{0 \leqslant k < n} |P_{n-1}(z_{k,n})| \tag{7.5}$$

holds for any polynomial $P_{n-1}(z)$ of degree at most $n-1$.

By Lemma 5.4, there exists $h(z) \in H^\infty(D_n)$ such that

$$h(z_{k,n}) = P_{n-1}(z_{k,n}), \qquad k = 0, 1, ..., n-1,$$

and

$$\|h(z)\|_{L^\infty(\Gamma_n)} \leqslant c \max |P_{n-1}(z_{k,n})| \leqslant c \, \|P_{n-1}(z)\|_{L^\infty(\Gamma)}. \tag{7.6}$$

For $z \in \Gamma$, we have

$$\begin{aligned} |h(z) - P_{n-1}(z)| &= \left| \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\omega(z)}{\omega(\zeta)} \frac{h(\zeta)}{\zeta - z} \, d\zeta \right| \\ &\leqslant \frac{1}{2\pi} |\omega(z)| \max_{\zeta \in \Gamma_n} \frac{1}{|\omega(\zeta)|} \, \|h\|_{L^\infty(\Gamma)} \int_{\Gamma_n} \frac{|d\zeta|}{|\zeta - z|}. \end{aligned} \tag{7.7}$$

By Lemma 6.1 and the maximum principle, we have

$$|\omega(z)| \max_{\zeta \in \Gamma_n} \frac{1}{|\omega(\zeta)|} \leqslant c.$$

From (2.8), as in the proof of Proposition 1 in [1], we have

$$\int_{\Gamma_n} \frac{|d\zeta|}{|\zeta - z|} \leqslant c \max \left\{ \log \frac{1}{d(z, \Gamma_n)}, 1 \right\} \leqslant c \log n.$$

Together with (7.6) and (7.7) we have (7.5).

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